$$\times \sum_{q_1^\circ} \int_{(2q_2^\circ-1)l}^{y} [R'(\mu)\, n(\eta, k) + 2cy\, R''(\mu)\, m(\eta, k)]\, \eta\, d\eta \qquad \text{(cont.)}$$

The derivatives of the Riemann function are taken in the argument $\mu = y^2 - \eta^2$; the integers $q_1^\circ$, $q_2^\circ$ follow from inequalities (3.5) considered at the boundary $y = 0$. The number of summands in the internal integrals of (4.5) differes at least by unity, hence a residual reflection wave may possibly be observed near a perpendicular wall. The function $F(t,k)$ defined by formulas (4.3)–(4.5), taken together, corresponds to the height of unsteady waves propagating along a perpendicular wall in a container. Expression (4.1) makes it possible to trace wave formation processes in the entire channel. The Fourier inversion serves to conclude our problem.

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# SELF-SIMILAR SPECTRA OF DECAYING TURBULENCE AT LARGE REYNOLDS AND PÉCLET NUMBERS

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We investigate the spectra of kinetic and thermal fluctuation energy of isotropic turbulence, neglecting the viscosity and molecular heat conduction. Elementary solutions of the corresponding spectral equations obtained here, enable us to investigate the properties of certain model spectra and a number of possible laws of decay in a simpler manner.

As we know [1], the spectral equations for the isotropic turbulence have the form

$$(\partial/\partial t + 2\nu k^2)\Phi(k, t) = -\partial/\partial k \; F(k, t) \tag{1}$$

$$(\partial/\partial t + 2\varkappa k^2)\; \Phi_{tt}(k, t) = -\partial/\partial k \; F_t(k, t) \tag{2}$$

where $k$ is the wave number, $t$ is time, $F$ and $F_t$ are the energy transfer functions, while $\Phi$ and $\Phi_{tt}$ are the fluctuation spectra defined by the equations

$$E(t) = \frac{1}{2}\langle u_i^2\rangle = \int_0^\infty \Phi(k, t)\, dk \tag{3}$$

$$E_t(t) = \langle T'^2\rangle = \int_0^\infty \Phi_{tt}(k, t)\, dk \tag{4}$$

We shall neglect the molecular effects ($\varkappa = \nu = 0$) and adopt the simplest expression for $F(k, t)$ which was proposed by Kovazhnyi

$$F(k, t) = \alpha^{-3/2} k^{5/2} \Phi^{3/2}(k, t) \tag{5}$$

together with its generalization to the case of the temperature field

$$F_t(k, t) = \alpha_t^{-1} \alpha^{-1/2} \Phi^{1/2}(k, t)\, \Phi_{tt}(k, t) \tag{6}$$

where $\alpha$ and $\alpha_t$ are universal constants, while $\varepsilon$ and $\varepsilon_t$ are the dissipation parameters.

Now, using the concept of the stability of large vortices [1], we shall assume that

$$\Phi(k, t) = \Lambda_m k^n, \quad k \to 0 \tag{7}$$

where $\Lambda_n = \Lambda(n)$ is a scale constant depending only on $n$. So far no complete agreement exists on the question of the value of the power index $n$ in (7). If we assume that Loitsianskii's invariant exists [2], then $n = 4$. On the other hand, the Kármán-Lin theory [3] gives the value $n = 1$, while Saffman [4] arrives at the value $n = 2$. Below, we shall consider $n$ as a free parameter.

Recently, Leith [5] put forward a new hypothesis on the self-similarity of the energy spectrum of degenerating turbulence at large Reynolds numbers $R$.

Using this hypothesis we can write

$$\Phi(k, t) = \lambda_n^3 t^{-2} \Phi_*(k_*), \quad k_* = k\lambda_n, \quad \lambda_n(t) = (\Lambda_n t^2)^{\frac{1}{n+3}} \tag{8}$$

where $\Phi_k(k_*)$ is a dimensionless function such, that

$$\Phi_*(k_*) = k_*^n \ (k_* \to 0), \quad \Phi_*(k_*) = a_n k_*^{-5/3} \quad (k_* \to \infty) \tag{9}$$

Determining $a_n$ in the usual manner [5] we obtain

$$a_n = \left[2 \frac{n+1}{n+3} E_n^*\right]^{2/3} \alpha, \quad E_n^* = \int_0^\infty \Phi_*(k_*)\, dk_* \tag{10}$$

We emphasise that $E_n^*$ is independent of $t$. Comparing (8)–(10) with the expression [2]

$$\Phi(k, t) = \alpha \varepsilon^{2/3}(t)\, k^{-5/3} \tag{11}$$

we easily obtain the possible laws of decay of the turbulence for various values of $n$; these are given in [5]. However, in order to close the Eq. (1), the author of [5] used the so-called diffusion approximation for $F(k, t)$, which does not permit obtaining the solution in the analytic form.

$$x^{\frac{n+3}{2}} = M^{-\frac{3n+9}{3n+6}} - M^{-1}, \quad \Psi(x) = x^n M^2(x) \tag{12}$$

$$\Phi_*(k_*) = \alpha^{\frac{3n}{n+3}} \Psi(x), \quad k_* = \alpha^{\frac{3}{n+3}} x$$

which can easily be solved for $n = 1$. When $k_* \to \infty$, (12) yields the following asymptotic representation:

$$\Phi_*(k_*) = \alpha^{(3n+5)/(n+3)} k_*^{-5/3}, \quad k_* \to \infty \tag{13}$$

From this, in accordance with (9), we obtain $a_n = \alpha^{(3n+5)/(n+3)}$. Comparing this with (10), we obtain

$$E_n^* = \int_0^\infty \Phi_*(k_*)\, dk_* = \frac{n+3}{2n+2} \alpha^{(3n+3)/(n+3)} \tag{14}$$

Let now the form of the temperature spectrum near the initial point $k = 0$ be

$$\Phi_{tt}(k, t) = \Lambda_{n_t} k^{n_t}, \quad k \to 0 \tag{15}$$

We know [2] that if the Korsin invariant exists, $n_t = 2$. For the sake of generality we shall assume that the parameter $n_t$ is free. Extending the Leith hypothesis [5] on the self-similarity of spectra in the process of degeneration, we can write

$$\Phi_{tt}(k, t) = \Lambda_{n_t} \lambda_n^{-n_t} \Phi_{tt}^*(k_*) \quad (k_* = k\lambda_n) \tag{16}$$

where $\lambda_n(t)$ is given by (8). The dimensionless function $\Phi_{tt}^*(k_*)$ must have the following asymptotes

$$\Phi_{tt}^*(k_*) = \begin{cases} k_*^{n_t} & (k_* \to 0) \\ a_{nn_t} k_*^{-5/3} & (k_* \to \infty) \end{cases} \tag{17}$$

Determining $a_{nn_t}$ in (17) in the manner used to define $a_n$ in (9) we obtain in accordance with [5]

$$a_{nn_t} = 2\frac{1+n_t}{3+n} \alpha_t \left(\frac{\alpha}{a_n}\right)^{1/2} E_{nn_t}^*$$

$$E_{nn_t}^* = \int_0^\infty \Phi_{tt}^*(k_*)\, dk_* \tag{18}$$

Table 1

| $n_t$ | $n = 0$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | $t^{-2/3}$ | $t^{-1/2}$ | $t^{-2/5}$ | $t^{-1/3}$ | $t^{-2/7}$ |
| 1 | $t^{-4/3}$ | $t^{-1}$ | $t^{-4/5}$ | $t^{-2/3}$ | $t^{-4/7}$ |
| 2 | $t^{-2}$ | $t^{-3/2}$ | $t^{-6/5}$ | $t^{-1}$ | $t^{-6/7}$ |

Table 1 gives the possible laws of decay of the temperature fluctuation intensity $E_t(t) \sim t^{-\sigma}$ corresponding to various values of $n$ near the initial point $k = 0$. The method of obtaining these relations is analogous to that used to obtain the laws governing the decay of the turbulence energy in [5].

Let us now turn our attention to Eq. (2). Using the expression (6), we obtain it in the following dimensionless form

$$k_* \frac{d\Phi_{tt}^*}{dk_*} - n_t \Phi_{tt}^* = -\frac{n+3}{2\alpha_t \sqrt{\alpha}} \frac{d}{dk_*} k_*^{5/2} \Phi_*^{1/2}(k_*) \Phi_{tt}^*(k_*) \tag{19}$$

This is linear in $\Phi_{tt}^*$, and its solution is

$$\Phi_{tt}^*(k_*) = k_*^{n_t} \exp\left[-\delta_n \int_0^{k_*} \frac{(n_t + 5/2)(x\Phi_*)^{1/2} + x^{3/2} d\sqrt{\Phi_*}/dx}{1 + \delta_n x^{3/2}\sqrt{\Phi_*}}\, dx\right] \tag{20}$$

$$\delta_n = (n+3)/2\alpha_t \sqrt{\alpha}$$

Here $\Phi_*(k_*)$ is the dimensionless kinetic energy spectrum obtained previously.

Let us consider an example. Inserting (9) into (20) and integrating, we obtain for very large $k_*$

$$\Phi_{tt}^*(k_*) = k_*^{n_t}(1 + \delta_n a_n^{1/2} k_*^{2/3})^{-1/2(5+3n_t)} \approx (\delta_n \sqrt{a_n})^{-1/2(5+3n_t)} k_*^{-5/3} \tag{21}$$

which agrees with (17) and moreover yields

$$a_{nn_t} = (\delta_n \sqrt{a_n})^{-1/2(5+3n_t)} \tag{22}$$

Finally, we obtain the equation

$$E_{nn_t}^* = \int_0^\infty \Phi_{tt}^*(k_*)\, dk_* = \frac{1}{1+n_t}\left(\frac{2\alpha_t}{n+3}\right)^{\frac{3(1+n_t)}{2}} \alpha^{\frac{-3(1+n)(1+n_t)}{2(n+3)}}$$

which is analogous to (14).

We must however keep in mind, that all spectra obtained here are closely connected with the Kovazhnyi approximations (5) and (6). When other approximations are employed, $E_n^*$ and $E_{nn_t}^*$ will be given by different formulas. At the same time, the laws of degeneration will follow directly only from the Leith's hypothesis (8) and from its generalization (16) to the case of a temperature field. We must also remember the fact that all results obtained here refer to the initial period of degeneracy of the isotropic turbulence, when the influence of viscosity and of molecular heat conduction can be neglected.

Simultaneous application of the Leith (8) and Kovazhnyi (5) hypotheses on the self-similarity of the energy spectrum in the initial stage of degeneracy of the isotropic turbulence and on the form of the energy transfer function followed by their generalization (16) and (6) to the case of a temperature field, leads to exact elementary solutions (12) and (20) for the corresponding spectral functions $\Phi(k, t)$ and $\Phi_{tt}(k, t)$. To the best of the author's knowledge, this is the first solution concerning the temperature spectrum to be offered. Having found the spectra, we can also determine the transfer functions $F(k, t)$ and $F_t(k, t)$, one-dimensional spectra $f(k, t)$ and $f_{tt}(k, t)$, the correlation functions $B(r, t)$ and $B_{tt}(r, t)$, the asymmetry coefficients $S(t)$ and $S_t(t)$ etc., and compare them with experimental data, as was done in [5–7]. We also hope that the relative simplicity of the mathematical analysis expounded above will also make it possible to solve the problem of stability of the solutions obtained, as exhaustively as required. Without this, of course, the solutions would have no practical significance. Until now, only some fragmentary results relevant to this problem were obtained using the Heisenberg approximation (see e. g. [8]). This, therefore, may form a subject for future investigations.

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